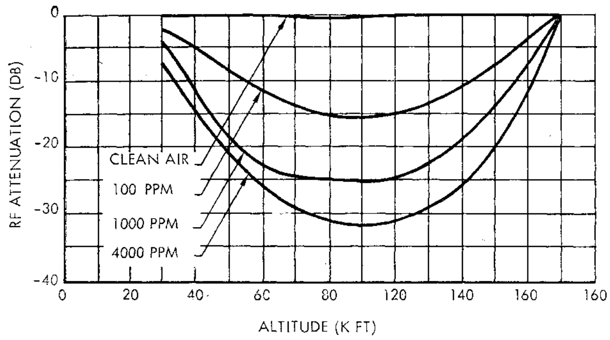

Fig. 2 RF attenuation for *C* band.

$(\omega_p/\omega)^2$. Let $X(Z)$ be a Dirac delta function

$$X(Z) = (\bar{X}L)\delta(Z) \tag{9}$$

$$\bar{X}L = \int_0^L X(Z)dZ \tag{10}$$

where $\bar{X}$ = mean electron density ratio.

Substituting Eqs. (9) and (10) into Eq. (8) and applying the boundary conditions, one obtains the simplified RF attenuation equation as follows:

$$T = 1 - R = \frac{1}{1 - i(N_0L/2)m_0X} = \frac{[1 + (\omega_c/\omega)A] + iA}{[1 + (\omega_c/\omega)A]^2 + A^2} \tag{11}$$

where

$$A = \frac{\pi}{\lambda}\frac{\left\{\int_0^L \left(\frac{\omega_p}{\omega}\right)^2 dz\right.}{\left.\left[1 + \left(\frac{\omega_c}{\omega}\right)^2\right]\right\}} \tag{12}$$

The attenuation equation in db becomes

$$\text{db} = 10 \log_{10} |T|^2 = 10 \log_{10}\left(\left\{\frac{[(1 + (\omega_c/\omega)A]}{[1 + (\omega_c/\omega)A]^2 + A^2}\right\}^2 + \left\{\frac{A}{[1 + (\omega_c/\omega)A]^2 + A^2}\right\}^2\right) \tag{13}$$

This equation was programed by Vicenti and Fletcher of Aerospace Corporation. The electron density profiles calculated by the method presented in the previous section are used as inputs for this program. The results of attenuation of an electromagnetic wave passing through the plasma sheath of a slender cone case are discussed in the next section.

Results and Conclusion

The attenuation of an RF signal generated by an 11°-cone flying a typical re-entry trajectory was studied. Figure 1 describes the electron density profiles at an altitude of 60,000 ft for various degrees of sodium contamination. The cone was considered at zero angle of attack with a velocity of 19,000 fps. Figures 2 and 3 describe the attenuation as a function of altitude for the various contaminant levels. For altitudes greater than 100,000 ft, a laminar boundary-layer analysis was used.[1] The pronounced effect of contaminants is obvious. Not only do they raise the peak electron concentration (Fig. 1), but the blowing significantly thickens the boundary layer, adding to the attenuation. For an ablating heat shield, the sodium contribution to RF attenuation completely predominates over the clean air contribution and is predictable by the previous simplified analysis.

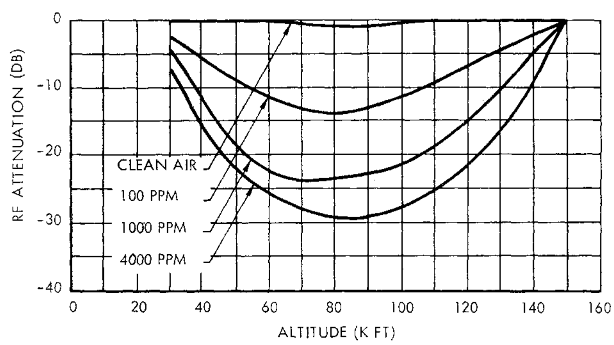

Fig. 3 RF attenuation for *X* band.

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Creep under Random Loading

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Introduction

The analysis of creep deformation under random loading should be based on a stress-strain relation that is valid for arbitrarily variable stress conditions. For linear viscoelastic materials, in the case of uniaxial stress, such a relation can be written in the form[1,2]

$$\epsilon(t) = \frac{1}{E}\left[\sigma(t) + \int_0^t \varphi(t - \theta)\sigma(\theta)d\theta\right] \tag{1}$$

where $\sigma(t)$ is the stress, $\epsilon(t)$ is the corresponding strain, and E and $\varphi(t)$ are a material constant and a material function, respectively. The constant E determines the instantaneous part of strain, which is proportional to the stress. The function $\varphi(t)$ is related to the rate of creep under constant stress, and the integral in Eq. (1) represents the time-dependent part of strain.

For metals at elevated temperatures, the following nonlinear stress-strain relation has been suggested and experimentally verified[3–5]:

$$\epsilon(t) = \frac{\sigma(t)}{E_M} + \frac{1}{\eta_K}\int_0^t e^{-(t-\theta)E_K/\eta_K} + \int_0^t \frac{\sigma(\theta)}{\eta_M(\theta)}d\theta \tag{2}$$

where E_M, E_K, and η_K are material constants, whereas $\eta_M(\theta)$ is a function of stress which may be assumed in the form

$$\eta_M(\sigma) = [B|\sigma|^{n-1}]^{-1} \tag{3}$$

with B and n being material constants. The first term on the right-hand side of Eq. (1) is the elastic part of strain, the

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second term is the primary (recoverable) creep, and the third term is the secondary (irrecoverable) creep.

Suppose now that the stress $\sigma(t)$ is a random function of time, given for $t > 0$. We shall assume that the random function $\sigma(t)$ is specified by the probability density function $f(\sigma)$, which for nonstationary processes depends on time, and by the correlation function $R_\sigma(t_1, t_2)$ defined as

$$R\sigma(t_1, t_2) \equiv \mathbf{E}[\sigma(t_1)\sigma(t_2)] \equiv \langle\sigma(t_1)\sigma(t_2)\rangle \tag{4}$$

where the symbols $\mathbf{E}[\ \ ]$ or $\langle\ \ \rangle$ denote the mean value.

Linear Viscoelastic Materials

Let us consider first the case of a linear viscoelastic material. The mean value of strain is, on the basis of (1),

$$\langle\epsilon(t)\rangle = \frac{1}{E}\left[\langle\sigma(t)\rangle + \int_0^t \varphi(t-\theta)\langle\sigma(\theta)\rangle d\theta\right] \tag{5}$$

where $\langle\sigma(t)\rangle$ is the mean value of stress.

The correlation function of strain $R_{\epsilon\xi}(t_1, t_2)$ is[6]

$$R_\xi(t_1, t_2) \equiv \langle\epsilon(t_1)\epsilon(t_2)\rangle = \frac{1}{E^2}\left[R_\sigma(t_1, t_2) + \int_0^{t_1} \varphi(t_1-\theta)R(t_2, \theta)d\theta + \int_0^{t_2} \varphi(t_2-\theta)R_\sigma(t_1, \theta)d\theta + \int_0^{t_1}\int_0^{t_2} \varphi(t_1-\theta_1)\varphi(t_2-\theta_2)R_\sigma(\theta_1, \theta_2)d\theta_1 d\theta_2\right] \tag{6}$$

By taking $t_1 = t_2 = t$, the mean square value of ϵ follows immediately

$$\langle\epsilon^2(t)\rangle = R_\epsilon(t, t) \tag{7}$$

For a linear viscoelastic material, the total stress may be written as a sum

$$\sigma(t) = \sigma_0(t) + \sigma_1(t) \tag{8}$$

where $\sigma_0(t)$ is a deterministic stress equal to the mean value of $\sigma(t)$, and $\sigma_1(t)$ is the remaining random part of $\sigma(t)$ (the mean value of $\sigma_1(t)$ is equal to zero). Accordingly, the strain $\epsilon(t)$ will consist of the part $\epsilon_0(t)$ corresponding to $\sigma_0(t)$, and the part $\epsilon_1(t)$ corresponding to $\sigma_1(t)$. With this decomposition of loading, the variance of ϵ is

$$\mathrm{var}(\epsilon) \equiv \langle\epsilon_1^2(t)\rangle = R_{\xi 1}(t, t) = \frac{1}{E^2}\left[R_{\sigma 1}(t, t) + 2\int_0^t \varphi(t-\theta)R_{\sigma 1}(t, \theta)d\theta + \int_0^t\int_0^t \varphi(t-\theta_1)\varphi(t-\theta_2)R_{\sigma 1}(\theta_1, \theta_2)d\theta_1 d\theta_2\right] \tag{9}$$

Let us note that if the stress $\sigma_1(t)$ is an ideal white noise, i.e., if

$$\begin{aligned} R_{\sigma 1}(t_1, t_2) &= \langle\sigma_1^2(t)\rangle \quad &&\text{for } t_1 = t_2 = t \\ R_{\sigma 1}(t_1, t_2) &= 0 &&\text{for } t_1 \neq t_2 \end{aligned} \tag{10}$$

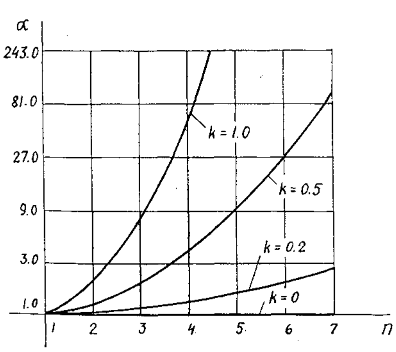

Fig. 1 Ratio $\langle\sigma^n\rangle/\langle\sigma\rangle^n$ vs n for three types of lognormal distribution.

then the integrals in the expression (9) vanish, and the variance of ϵ is

$$\mathrm{var}(\epsilon) \equiv \langle\epsilon_1^2(t)\rangle = \frac{1}{E^2}\langle\sigma_1^2(t)\rangle \tag{11}$$

In some cases, the function $R\sigma_1(t_1, t_2)$ may be approximated by

$$R_{\sigma 1}(t_1, t_2) \approx r\delta(t_1 - t_2) \tag{12}$$

where δ is the Dirac function and r is

$$r = r(t_1) = \int_{-\infty}^{\infty} R_{\sigma 1}(t_1, t_2)dt_2 = r(t_2) = \int_{-\infty}^{\infty} R_{\sigma 1}(t_1, t_2)dt_1$$

With (12), the expression (10) becomes

$$\mathrm{var}(\epsilon) \equiv \langle\epsilon_1^2(t)\rangle = \frac{1}{E^2}\left[\langle\sigma_1^2(t)\rangle + 2r(t)\varphi(0) + \int_0^t r(\theta)\varphi^2(t-\theta)d\theta\right] \tag{13}$$

Nonlinear Materials

We shall consider now the nonlinear material with the stress-strain relation (2) and (3). The following expression results for the mean value of strain

$$\langle\epsilon(t)\rangle = \frac{\sigma(t)}{E_M} + \frac{1}{\eta_K}\int_0^t \epsilon^{-(t-\theta)E_K/\eta_K}\langle\sigma(\theta)\rangle d\theta + \int_0^t B\mathbf{E}[|\sigma(\theta)|^{n-1}\sigma(\theta)]d\theta \tag{14}$$

The first two terms represent the elastic strain and the primary creep. The third term represents the secondary creep. To simplify our discussion, we shall assume that either the stress $\sigma(t)$ is always positive or always negative, or n is an odd integer. The mean value of the rate of secondary creep is then

$$\mathbf{E}[\epsilon_{II}(t)] \equiv \langle\epsilon_{II}(t)\rangle = B\mathbf{E}[\sigma^n(\theta)] = B\int_{-\infty}^{\infty} \sigma^n f(\sigma)d\sigma \tag{15}$$

where $f(\sigma)$ is the probability density function of the stress σ. Thus, the mean rate of secondary creep is related to the nth moment of the probability density function $f(\sigma)$ (and not to the nth power of the mean value of σ).

The mean square value of strain is

$$\langle\epsilon^2(t)\rangle = \frac{\langle\sigma^2(t)\rangle}{E_M} + \frac{2}{E_K\eta_K}\int_0^t e^{-(t-\theta)E_K/\eta_K}\langle\sigma(t)\sigma(\theta)\rangle d\theta + \frac{1}{\eta_K^2}\int_0^t\int_0^t e^{-[(t-\theta_1)+(t-\theta_2)]E_K/\eta_K}\langle\sigma(\theta_1)\sigma(\theta_2)\rangle d\theta_1 d\theta_2 + 2\int_0^t \frac{B}{E_M}\langle\sigma^n(\theta)\sigma(t)\rangle d\theta + 2\int_0^t\int_0^t \frac{B}{\eta_K} e^{-(t-\theta)E_K/\eta_K} \cdot \langle\sigma^n(\theta_2)\sigma(\theta_1)\rangle d\theta_1 d\theta_2 + \int_0^t\int_0^t B^2\langle\sigma^n(\theta_1)\sigma^n(\theta_2)\rangle d\theta_1 d\theta_2 \tag{16}$$

The preceeding expression is rather complicated; it contains several terms, and the correlation functions of the type $\langle\sigma^n(t_1)\sigma(t_2)\rangle$ and $\langle\sigma^n(t_1)\sigma^n(t_2)\rangle$ have to be used. Certain simplifications may be introduced in the same manner as for linear viscoelastic materials, i.e., by replacing the actual forms of the correlation functions with simpler expressions shown in Eq. (12). Moreover, for longer times, the secondary creep becomes much larger than the elastic strain and the primary creep. Therefore

$$\langle\epsilon^2(t)\rangle \approx \langle\epsilon_{II}^2(t)\rangle = B^2\int_0^t\int_0^t \langle\sigma^n(\theta_1)\sigma^n(\theta_2)\rangle d\theta_1 d\theta_2 \tag{17}$$

Conclusions

The results presented for linear viscoelastic materials are not unexpected. The mean value of strain is directly related to the mean value of stress. The correlation function of strain follows from the type of calculation commonly used in the theory of random processes in linear systems. The effect of the temporal correlation of the random stress $\sigma(t)$ is evident from (12) and (13). For weakly correlated random process $\sigma(t)$, the value of r is small, and consequently the variance of the strain ϵ also small. Thus, the variance of strain depends on the variance and the temporal correlation of stress.

The nonlinear component of creep strain, denoted by ϵ_{II} is more complicated. Its mean value depends on the type of the probability density function of stress $f(\sigma)$ and it follows from Eq. (14). In some cases [see Eq. (15)], the mean value of nonlinear creep rate is proportional to $\langle\sigma^n\rangle$, i.e., the nth moment of $f(\sigma)$ with respect to the origin, where n is the power in the nonlinear stress-strain relation (2) and (3). If only the mean value of stress were used in Eq. (2), the resulting creep rate would be proportional to $\langle\sigma\rangle^n$. The ratio

$$\alpha \equiv \langle\sigma^n\rangle/\langle\sigma\rangle^n \tag{18}$$

(which is also equal to the ratio of the correct mean value of the nonlinear creep rate and the incorrect value obtained by using $\langle\sigma\rangle$) shows the effects of the nonlinearity of the material and the scatter of the applied stress.

The case of lognormal distribution of the stress has been investigated in some detail. The nth moment about origin is[7]

$$\langle\sigma^n\rangle = e^{nm+n^2c^2/2} \tag{19}$$

where m and c are the parameters of the distribution; the mean $\langle\sigma\rangle$ is

$$\langle\sigma\rangle = e^{m+c^2/2} \tag{20}$$

We can write therefore the following formula for the ratio α

$$\alpha \equiv \langle\sigma^n\rangle/\langle\sigma\rangle^n = e^{(n^2-n)c^2/2} \tag{21}$$

The parameter c can be determined from the relation

$$k^2 = e^{c^2} - 1 \tag{22}$$

where

$$k^2 = \text{var}(\sigma)/\langle\sigma\rangle^2 = \langle(\sigma - \langle\sigma\rangle)^2\rangle/\langle\sigma\rangle^2 \tag{23}$$

i.e., k is the ratio of the standard deviation to the mean value of stress.

Figure 1 shows the values of α plotted as functions of n for the following four values of k: $k = 0, 0.1, 0.2, 0.5, 1.0$. Only for $n = 1$ (linear material) or $k = 0$ (deterministic stress) it is $\alpha = 1$. In all the other cases, the effect of nonlinearity and the effect of the scatter of stress are strongly pronounced.

In the case of the rectangular density function given by

$$\begin{aligned} f(\sigma) &= 1/2\langle\sigma\rangle \text{ for } 0 < \sigma < 2\langle\sigma\rangle \\ f(\sigma) &= 0 \text{ for } \sigma < 0 \text{ and } \sigma > 2\langle\sigma\rangle \end{aligned} \tag{24}$$

the ratio α becomes

$$\alpha = 2^n/(n + 1) \tag{25}$$

For metals at elevated temperatures, the value of n is usually between 3 and 5; the corresponding values of α are $2 < \alpha < 5.33$.

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